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Transverse conductivity in non-ideal fiber composite geometries

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Abstract—An ideal fiber composite is defined as one with parallel circular fibers forming a regular array. We consider how the thermal conductivity transverse to the overall fiber direction is affected by departures from the ideal geometry, such as irregular stacking pattern, misalignment of fibers and unequal or non-circular cross sections. Illuminating examples are discussed, quantitatively and qualitatively. In particular, we derive an effective-medium result for slightly misaligned fibers. It is found that for fiber concentrations not close to a percolation threshold, the effect of non-ideal geometry is normally too small (< 1% in many realistic examples) to be of any practical importance.

1. INTRODUCTION

WE CONSIDER conduction perpendicular to the fiber orientation in a fiber composite. By an ideal fiber geometry we mean that the fibers are infinitely long, of equal circular cross section and stacked in a regular array, such as to give overall isotropic conductivity transverse to the fibers. The purpose of this paper is to discuss systematically how the conductivity is affected by deviations from the ideal geometry, such as irregularities in the stacking, misalignments of the fibers and deviations from equal and circular cross sections. Through the presentation of a number of representative examples, which are treated quantitatively or qualitatively, we shall compare the importance of various types of non-ideal geometric features.

Several recent papers have dealt with the transverse conductivity of fiber composites and results we shall require are implicitly contained in some of them [1–9]. For general aspects of transport in inhomogeneous materials, the reader is referred to recent reviews [10–12]. Our geometry is essentially equivalent to that with disks distributed in a plane. Some recent works refer to such two-dimensional systems with random distributions of disks [13–15] and to the effective conductivity of regular arrays of cylinders with square or circular cross sections [1, 2, 16]. The results obtained in this paper apply also to the electrical conductivity, the dielectric constant and the magnetic permeability.

Let the conductivities of the fibers and the matrix be k_1 and k_2 and the effective transverse conductivity of the composite be k_{eff}. The contrast between the fiber and the matrix is defined as the conductivity ratio $\alpha \equiv k_1/k_2$. The volume fractions of the fibers and the matrix are f_1 and $1-f_1$, respectively. For a certain stacking geometry, the percolation threshold f_c is that volume fraction for which fibers begin to touch so as to form a continuous path through the composite.

It will be assumed that the fibers have the higher conductivity, i.e. the contrast $\alpha > 1$. Results for $\alpha < 1$ follow from the reciprocity theorem, which holds in two dimensions for the interchange of phases without altering the phase boundaries [17, 18]

$$k_{\text{eff}}(k_1, f_1; k_2, 1-f_1)k_{\text{eff}}(k_2, f_1; k_1, 1-f_1) = k_1 k_2. \quad (1)$$

2. PERIODIC AND RANDOM STACKING OF FIBERS

In the dilute limit, $f_1 \ll 1$, there is negligible interaction between the fibers and the effective conductivity is independent of the stacking geometry. For parallel fibers with randomly oriented elliptical cross sections, one then has [10]

$$k_{\text{eff}} = k_2 - f_1\left(\frac{k_2-k_1}{2}\right)\sum_{i=1}^{2}\frac{k_2}{k_2+A_i(k_1-k_2)}. \quad (2)$$

Here $A_1 = a/(a+b)$, $A_2 = b/(a+b)$, where a and b are the semiaxes of the elliptical cross section.

When going beyond the dilute limit, k_{eff} must be calculated numerically. There are five symmetrically different ways of stacking parallel cylinders in regular arrays. Two of them, with hexagonal and square patterns respectively, give isotropic conductivities in the plane perpendicular to the fiber axis. Several authors

NOMENCLATURE

A_1, A_2 semiaxes ratios
a, b semiaxes
D specimen size
f phase volume fraction
f_1 fiber volume fraction
K normalized thermal conductivity
K_e conductivity tensor
k thermal conductivity
k_1 conductivity of fiber
k_2 conductivity of matrix
L fiber length
R, r fiber radius
$\mathbf{r}$ position vector
x, y, z, x', y', z' Cartesian coordinates.

Greek symbols
α conductivity ratio, fiber to matrix k_1/k_2
θ fiber tilt angle
$\Delta\theta$ r.m.s. fiber tilt angle.

Subscripts
1 fiber
2 matrix
3 clustered region, considered as a separate phase
c percolation threshold
chq composite with checkerboard geometry
clu composite with clustering
e⊥ effective conductivity transverse to fiber axes
eff effective property of composite
e effective property of composite
HS Hashin–Shtrikman coated cylinder geometry
hex composite with fibers in hexagonal stacking pattern
m summation index, 1, 2, ...
ran composite with fibers in random lateral stacking
sqr composite with fibers in a square stacking pattern.

[1, 2, 16] have considered the square lattice array. Let the fiber radius be R and the lattice parameter be a. The volume fraction of fibers is $f = \pi R^2/a^2$ and the percolation threshold is $f_c = \pi/4 \approx 0.785$. Perrins *et al.* [1] calculated the effective conductivity, k_{sqr}, of this system for some contrasts α. We have performed similar calculations (as described in ref. [16] and extended in the present work) to obtain k_{chq} for other α-values of interest. Perrins *et al.* [1] also calculated the effective conductivity, k_{hex}, for a hexagonal stacking of circular fibers. In that case, the percolation threshold is $f_c = \pi/(2\sqrt{3}) \approx 0.907$.

Kim and Torquato [14, 15] calculated the effective transverse conductivity, k_{ran}, for a random 'equilibrium' distribution of parallel circular fibers. The distribution was essentially generated by starting from the square-lattice array, and then performing a long sequence of small random movements of the fibers, while keeping them parallel. By construction the calculations are limited to $f_1 < \pi/4$. Figure 1 shows k_{sqr}, k_{hex} and k_{ran} versus $\log(\alpha)$ for fiber volume fractions $f_1 = 0.2$ and $f_1 = 0.4$, respectively, and with k_{chq} obtained from the numerical work referred to above.

3. CLUSTERING

3.1. *Volume fractions and conductivities on a mesoscale*

We shall distinguish between three length scales; a microscale which is of the order of the fiber radius R, a macroscale of the order of the specimen size D, and between them a mesoscale L, such that $R \ll L \ll D$. Clustering can be described by a properly averaged fiber density $f(\mathbf{r})$. More specifically, consider a certain plane perpendicular to the average fiber axes. Let $f(\mathbf{r})$ be the area fraction of fibers in a disk of radius l centered at $\mathbf{r}$. For increasing l, $f(\mathbf{r})$ will first fluctuate on the microscale, then take a well-defined value (although with some 'noise') on the mesoscale, and eventually approach the overall volume fraction $f_1 \equiv \langle f(\mathbf{r}) \rangle$ on the macroscale. Here $\langle \ldots \rangle$ denotes a spatial average over the entire specimen. Similarly, we can define a local conductivity on the mesoscale in the composite, $k_{loc}(\mathbf{r})$.

3.2. *Weak clustering*

When the local conductivity k does not vary much throughout the specimen, the total effective conductivity, in two dimensions, can be written [19, 20]

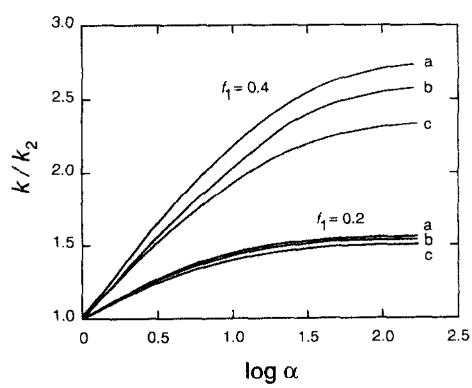

FIG. 1. The influence of stacking geometries (hexagonal, square and random) of circular fibers on the transverse effective conductivity, expressed as ratios $k_{hex}/k_2 \approx k_{sqr}/k_2$ (b) and k_{ran}/k_2 (a), plotted vs $\log(\alpha)$, for fiber volume concentrations $f_1 = 0.2$ and 0.4, respectively. Here α is the conductivity ratio between fiber and matrix. The figure also gives the ratio k_{chq}/k_2 (curves c) for square fiber cross sections in a checkerboard geometry.

$$k_{\text{eff}} = \langle k \rangle \left\{ 1 - \left(\frac{1}{2}\right) \frac{\langle (k - \langle k \rangle)^2 \rangle}{\langle k \rangle^2} \right\}. \quad (3)$$

If we expand $k_{\text{loc}}(\mathbf{r})$ in powers of $\varepsilon(\mathbf{r}) \equiv f(\mathbf{r}) - \langle f \rangle$, $k_{\text{loc}} = k_0 + \varepsilon k' + (\varepsilon^2/2)k''$, and keep terms to order ε^2 we get

$$k_{\text{eff}} = k_0\{1 - \tfrac{1}{2}\langle \varepsilon^2 \rangle [(k'^2/k_0^2) - (k''/k_0)]\}. \quad (4)$$

Here k_0, $k' = \partial k/\partial f$, and $k'' = \partial^2 k/\partial f^2$ are evaluated for $f = \langle f \rangle$. We define a quantity $B(\alpha)$,

$$B = (k'^2/k_0^2) - (k''/k_0) \quad (5)$$

so that

$$k_{\text{eff}} = k_0\{1 - \tfrac{1}{2}\langle \varepsilon^2 \rangle B(\alpha)\}. \quad (6)$$

For $f_1 < 0.4$ it is a very good approximation to write [1]

$$k_{\text{sqr}} \approx k_{\text{hex}} \approx k_2 \left\{ 1 - \frac{2f_1}{f_1 + T} \right\} \quad (7)$$

where

$$T = \frac{1 + \alpha}{1 - \alpha}. \quad (8)$$

Then, from (7),

$$B = \frac{4Tf_1}{(T - f_1)^2 (T + f_1)^2}. \quad (9)$$

We note that $B < 0$. For infinite contrast, $\alpha = \infty$ and $T = -1$, one has $B = -4f_1/(1 - f_1^2)^2$, which yields $B = -0.868$ and $B = -2.27$ for $f_1 = 0.2$ and 0.4, respectively. As an illustrating example, let $f(\mathbf{r})$ vary sinusoidally in two perpendicular directions in the plane perpendicular to the fibers, between the extreme values $f_{\text{min}} = 0.15$ and $f_{\text{max}} = 0.25$, with $\langle f \rangle = 0.2$. Then $\langle \varepsilon^2 \rangle = 1/1600$, and the effect of a fluctuating $f(\mathbf{r})$ on k_{eff} is negligible, even in the case of $\alpha = \infty$.

3.3. *Strong clustering*

We next turn to strong clustering. As a specific example (Fig. 2) consider circular fibers clustered into a square array of two-phase circular 'fibers' which can now be regarded as a single phase i, with volume fraction f_i. Inside phase i, the original fibers also form a square array. Conservation of the total volume fraction f_1 of the original fibers requires that their volume fraction in phase i is $f_3 = f_1/f_i$. This geometry is chosen because it allows a numerical account of the effect of clustering, relying on the work on k_{sqr}. (Cf. similar arguments for the clustering of spheres [21].) The overall effective conductivity of the clustered composite is $k_{\text{clu}} = k_{\text{sqr}}(k_{\text{sqr}}(f_3), f_i, k_2, \alpha)$. Figure 3 shows $1 - k_{\text{clu}}/k_{\text{sqr}}$, as a function of $\log \alpha$ for clustering such that $f_1 = 0.1, f_3 = 0.25$ and $f_i = 0.4$.

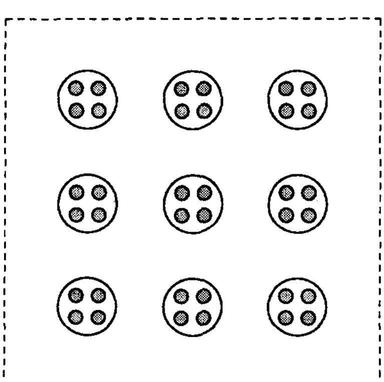

FIG. 2. A schematic illustration of a model for fiber clustering.

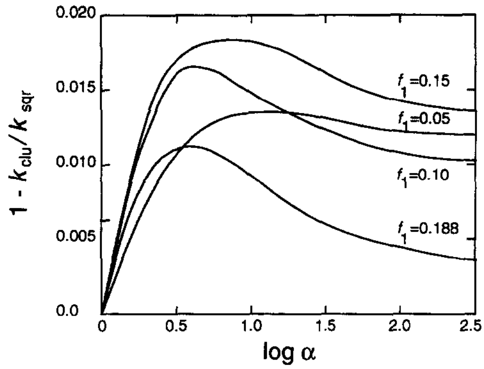

FIG. 3. The effective conductivity of a clustered composite, k_{clu}, compared with stacked in a uniform square pattern, k_{sqr}, here expressed as $1 - k_{\text{clu}}/k_{\text{sqr}}$, for several volume fractions f_1 of the fiber, and with $f_3 = 0.25$.

In an extreme clustering we could imagine that the fibers in the bundles that define the phase i are completely fused together, so as to form new circular fibers consisting only of phase 1. Since k_{sqr} depends only on the volume fraction and not on the fiber radius itself, such a clustering would leave the effective conductivity of the composite unchanged.

4. NON-ALIGNMENT OF FIBERS

4.1. *Long fibers with weak non-alignment*

Let the fibers be much longer than their diameter. Then it is an excellent approximation to obtain the effective transverse conductivity by a parallel coupling of the conductivities of a sequence of 'slices',

$$k_{\text{eff}} = (1/L) \int_0^L k_p(x)\, dx, \quad (10)$$

where $k_p(x)$ is the effective conductivity per length of a 'slice', perpendicular to the fiber axes of the aligned fibers, and x is the position along those fiber axes.

Consider a special case of non-alignment obtained as follows. Let the fibers extend from $x = 0$ to $x = L$ and let all their centers at $x = L/2$ form a regular square lattice. The fibers are pivoted around their centers at $x = L/2$, so that the end points at $x = L$ are moved to the points of the two-dimensional equilibrium distribution of disks considered by Kim and Torquato [14, 15] and referred to above. The fact that geometric constraints for the rigid bars may make a few such connections impossible is of no importance for our qualitative conclusions. Now, from equation (10),

$$k_{\rm ran} > k_{\rm eff} > k_{\rm sqr} \tag{11}$$

since $k_p(L/2) = k_{\rm sqr} < k_{\rm ran} = k_p(0) = k_p(L)$. Hence, non-alignment starting from a square lattice configuration of fibers increases the effective conductivity. If we instead start from a hexagonal lattice configuration, $k_{\rm eff}$ would also increase since we noted above that $k_{\rm hex} \approx k_{\rm ran}$ for moderate f_1.

4.2. *Effective-medium theory*

Consider a single fiber embedded in an effective medium consisting of the matrix and all other fibers. The axis of the single fiber forms an angle θ with the overall fiber direction. We let θ vary according to a normalized distribution function

$$g(\theta) = \frac{2m+1}{4\pi}\cos^{2m}\theta, \quad 0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \varphi \leqslant 2\pi. \tag{12}$$

The limit $m \to \infty$ corresponds to parallel fibers and $m = 0$ to entirely disordered fiber axes directions. Using ordinary effective-medium arguments for fibers modelled as very prolate ellipsoids and averaging over all θ in equation (12) gives expressions (Appendix 1) from which the normalised effective conductivities $K_{e\parallel} = k_{e\parallel}/k_2$ in the longitudinal direction and $K_{e\perp} = k_{e\perp}/k_2$ in the transverse directions can be obtained. We are primarily interested in nearly aligned fibers, i.e. large m. From equation (A10), and in a series expansion which retains terms of order $1/m$, one obtains after some algebra, and with $q = k_{e\perp}(\infty)/k_2$,

$$\frac{k_{e\perp}(m)}{k_{e\perp}(\infty)} - 1 = \frac{f_1}{2mq}(\alpha - q)\left[\frac{(1-n)q+\alpha n}{(1-n_x)q+\alpha n_x}\right] \times \left[\frac{[(n_x-n)q+(n_x+n)\alpha](2q+1)^2}{f_1\alpha(2q+1)^2+9f_2[(1-n)q+\alpha n]^2}\right]. \tag{13}$$

The mean deviation $\Delta\theta$ from $\theta = 0$ is given by

$$\Delta\theta = 2^m \frac{m!}{(2m+1)!!}. \tag{14}$$

Figure 4 shows $[K_{e\perp}(\Delta\theta)/K_{e\perp}(0)] - 1$ from equation (13) but expressed and plotted as a function of $\Delta\theta$, calculated from equation (14) for a prolate ellipsoid with semiaxes ratio $a/b = 10$, fiber volume fraction $f_1 = 0.2$ and conductivity ratios $\alpha = 10$, 10^3, 10^5 and ∞. Clearly, the effect of non-alignment on the transverse conductivity is small only for not very large contrasts α and for small misalignments $\Delta\theta$. This is expected since non-aligned, highly conducting long fibers may touch and form a percolating path of high conductivity.

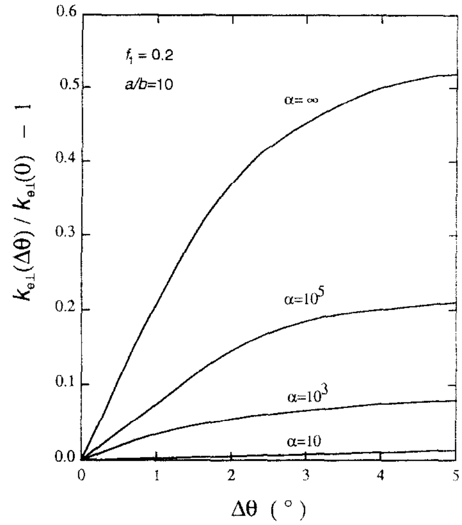

FIG. 4. The effect of fiber non-alignment on the transverse effective conductivity. The figure shows $[k_{e\perp}(\Delta\theta)/k_{e\perp}(0)] - 1$ as a function of the r.m.s. non-alignment angle $\Delta\theta$, for fiber volume fraction $f_1 = 0.2$ and several conductivity ratios α, and with fibers modelled as ellipsoids with aspect ratio 10.

5. NON-IDEAL FIBER CROSS SECTION

5.1. *Dispersion in fiber radius*

In the Hashin–Shtrikman coated-cylinder geometry (Fig. 5) there is an exact expression for the effective conductivity [10];

$$k_{\rm HS} = k_2 + f_1[1/(k_1-k_2) + f_2/(2k_2)]^{-1}. \tag{15}$$

Each fiber and its coating layer have the same volume fractions as in the entire material. The coated cylinders can take all sizes so as to allow a space-filling packing. $k_{\rm HS}$ also is a lower bound to $k_{\rm eff}$ if nothing is known about the geometry of the system except the volume fraction f_1 and that the overall conductivity is isotropic. For small f_1, $k_{\rm HS}$ agrees (to leading order in f_1) with the expression (2) when $a = b$, i.e. a dilute suspension of circular cylinders. Miller and Torquato [9] and others have discussed improved lower bounds which contain information on the spatial correlation function for the fibers through a parameter ζ_2.

Consider circular fiber cross sections, but with vary-

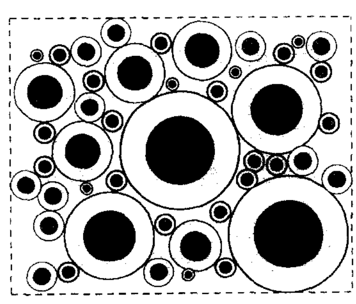

FIG. 5. Schematic illustration of the Hashin–Shtrikman coated-cylinder geometry. The coated cylinders have all sizes, so that the entire space can be filled with them.

ing radii. Miller and Torquato [9] calculated how the effective conductivity k_{ran} for a random distribution of aligned fibers is changed when the fibers are given a dispersion in their radii. We follow their approach, but consider equal numbers of two kinds of fibers, with radii r_1 or r_2. The total volume fraction is conserved so that $r_1^2+r_2^2 = 2r_0^2$, where r_0 is the radius of the fibers in the case to compare with. The variation in fiber radius is assumed to be small, i.e. $\Delta = |r_2-r_1|/r_0 \ll 1$. We find that the dispersion in fiber radius gives

$$\zeta_2 = \tfrac{1}{3}f_2(1+\tfrac{1}{18}\Delta^2). \tag{16}$$

Hence, a variation in fiber radius by $\pm 10\%$, i.e. $\Delta = 0.1$, gives an entirely negligible correction to ζ_2 in equation (16). We conclude that such a dispersion has no significant effect on k_{eff}.

5.2. *Elliptical and square fiber cross section*

Consider the dilute limit of parallel fibers with elliptical cross sections, equation (3), and let the aspect ratio be $a/b = (1+\delta)/(1-\delta)$. The largest effect of a non-circular cross section arises for the case of extreme contrast, $\alpha = \infty$. Then, to leading order in the small quantity δ, equation (2) takes the form

$$k_{eff} = k_2[1+2f_1(1+\delta^2/2)]. \tag{17}$$

As an example, let $f_1 = 0.2$ and $a/b = 5/4$. Then k_{eff} in equation (17) is larger than k_{eff} of circular fibers by only 0.2%.

We next turn to square fibers. In a previous work [16] we calculated the effective transverse conductivity k_{chq} when the fibers are stacked in a square array such that their cross sections form a checkerboard at the percolation threshold $f_{chq} = 0.5$ (Fig. 6). Extending those numerical calculations, we compare square and circular fiber cross sections and plot $k_{chq}(f)/k_{sqr}(f)-1$ vs $\log(\alpha)$ in Fig. 7.

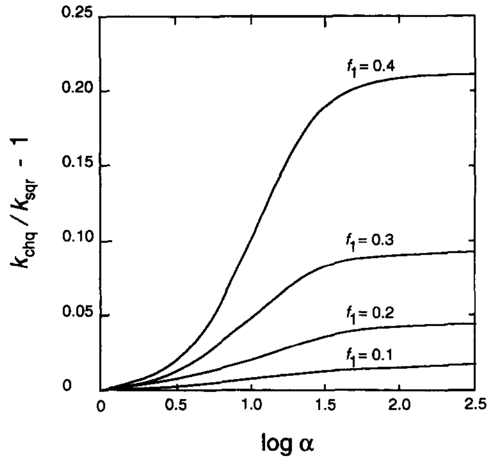

FIG. 7. Comparison between the effective conductivities k_{chq} and k_{sqr}, referring to square and circular fiber cross sections, with fibers stacked in a square lattice, for varying $\log(\alpha)$ and some fiber volume fractions f_1.

6. DISCUSSION AND CONCLUSIONS

In Sections 2–5 we have considered various types of deviations from ideal fiber composite geometries, and their effect on the transverse thermal conductivity k_{eff}. The effect is of course largest when the ratio α between the conductivities of the fibers and the matrix is infinite (or zero). However, even in that case the non-ideal conditions often lead to corrections that are negligible from a practical point of view. We can exemplify this by taking a conductivity ratio $\alpha = \infty$ and a fiber volume fraction $f_1 = 0.2$. Then a fiber concentration slowly varying as $f_1 = 0.20 \pm 0.05$, a dispersion in fiber radius by $\pm 10\%$ or a fiber cross section varying from circular to elliptic with semiaxes ratios 5/4, all affect the transverse conductivity by less than 1%. Only when the volume fraction or the fiber cross section is such that one approaches a percolation threshold does the non-ideal fiber geometry significantly affect the transverse conductivity. That may also be the case for long non-aligned fibers. Only for moderate contrasts ($\alpha < 10$) is the effect of a non-alignment given by $\Delta\theta = 6°$ less than 1%.

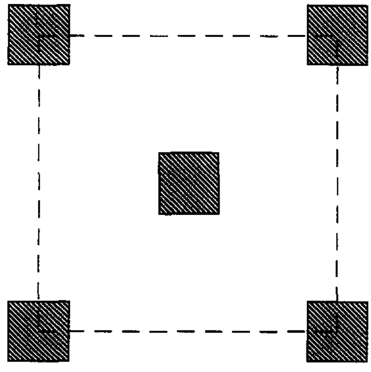

FIG. 6. A checkerboard geometry.

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REFERENCES

1. W. T. Perrins, D. R. McKenzie and R. C. McPhedran, Transport properties of regular arrays of cylinders, *Proc. R. Soc. Lond. Ser. A* **369**, 207–225 (1979).
2. G. W. Milton, R. C. McPhedran and D. R. McKenzie, Transport properties of arrays of intersecting cylinders, *Appl. Phys.* **25**, 23–30 (1981).
3. A. S. Sangani and C. Yao, Transport processes in random arrays of cylinders. I. Thermal conduction, *Phys. Fluids* **31**, 2426–2434 (1988).
4. S. Torquato and F. Lado, Bounds on the effective transport and elastic properties of a random array of cylindrical fibers in a matrix, *J. Appl. Mech.* **55**, 347–354 (1988).
5. K. Muralidhar, Equivalent conductivity of a hetero-

geneous medium, *Int. J. Heat Mass Transfer* **33,** 1759–1766 (1990).
6. R. Tao, Z. Chen and P. Sheng, First-principles Fourier approach for the calculation of the effective dielectric constant of periodic composites, *Phys. Rev. B* **41,** 2417–2420 (1990).
7. Y. C. Chiew, Effective conductivity of two-phase materials consisting of long parallel cylinders, *J. Appl. Phys.* **67,** 1684–1688 (1990).
8. S. M. Grove, A model of transverse thermal conductivity in unidirectional fiber-reinforced composites, *Comp. Sci. Tech.* **3,** 199–209 (1990).
9. C. A. Miller and S. Torquato, Improved bounds on elastic and transport properties of fiber-reinforced composites: effect of polydispersivity in fiber radius, *J. Appl. Phys.* **69,** 1948–1955 (1991).
10. G. Grimvall, *Thermophysical Properties of Materials*, Chap. 13. North-Holland, Amsterdam (1986).
11. S. Torquato, Random heterogeneous media: microstructure and improved bounds on effective properties, *Appl. Mech. Rev.* **44,** 37–76 (1991).
12. S. Torquato, Thermal conductivity of disordered heterogeneous media from the microstructure, *Rev. Chem. Engng* **4,** 151–204 (1987).
13. P. P. Durand and L. H. Ungar, Application of the boundary element method to dense dispersions, *Int. J. Numer. Methods Engng* **26,** 2487–2501 (1988).
14. I. C. Kim and S. Torquato, Determination of the effective conductivity of heterogeneous media by Brownian motion simulation, *J. Appl. Phys.* **68,** 3892–3903 (1990).
15. I. C. Kim and S. Torquato, First-passage-time calculation of the conductivity of continuum models of multiphase composites, *Phys. Rev. A* **43,** 3198–3201 (1991).
16. Bao Ke-da, J. Axell and G. Grimvall, Electrical conduction in checkerboard geometries, *Phys. Rev. B* **41,** 4330–4333 (1990).
17. J. B. Keller, A theorem on the conductivity of a composite medium, *J. Math. Phys.* **5,** 548–549 (1964).
18. K. S. Mendelson, Effective conductivity of two-phase material with cylindrical phase boundaries, *J. Appl. Phys.* **46,** 917–918 (1975).
19. C. Herring, Effect of random inhomogeneities on electrical and galvanomagnetic measurements, *J. Appl. Phys.* **31,** 1939–1953 (1960).
20. L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, p. 46. Pergamon, Oxford (1960).
21. W. T. Doyle and I. S. Jacobs, Effective cluster model of dielectric enhancement in metal–insulator composites, *Phys. Rev. B* **42,** 9319–9327 (1990).
22. R. Landauer, The electrical resistance of binary metallic mixtures, *J. Appl. Phys.* **23,** 779–784 (1952).

APPENDIX

Using the effective medium theory [22] for three-dimensional elliptic inclusions (conductivity k_1) in a matrix (conductivity k_2; contrast $\alpha = k_1/k_2$), one first expresses the conductivity tensor K_e in (x', y', z') coordinates (cf. Fig. 8) as

$$\begin{pmatrix} K_{ex'x'} & K_{ex'y'} & K_{ex'z'} \\ K_{ey'x'} & K_{ey'y'} & K_{ey'z'} \\ K_{ez'x'} & K_{ez'y'} & K_{ez'z'} \end{pmatrix} = \begin{pmatrix} K_{e\perp}+QS_{x'x}^2 & QS_{x'x}S_{y'x} & QS_{x'x}S_{z'x} \\ QS_{y'x}S_{x'x} & K_{e\perp}+QS_{y'x}^2 & QS_{y'x}S_{z'x} \\ QS_{z'x}S_{y'x} & QS_{z'x}S_{y'x} & K_{e\perp}+QS_{z'x}^2 \end{pmatrix}. \quad \text{(A1)}$$

Here $S_{x'x} = \cos\theta$, $S_{y'x} = \sin\theta\cos\varphi$, $S_{z'x} = \sin\theta\sin\varphi$, $K_{e\parallel} = K_{ex}$, $K_{e\perp} = K_{ey} = K_{ez}$. Further, $K_{e\parallel} = 1+f_1\beta_x(\alpha-1)$, $K_{e\perp} = 1+f_1\beta(\alpha-1)$, $\beta_x = 1/[1+n_x(\alpha-1)]$, $\beta = 1/[1+n(\alpha-1)]$, $Q = K_{e\parallel}-K_{e\perp}$, and n_x and $n = n_y = n_z$ are the depolarization factors in the longitudinal and transverse directions,

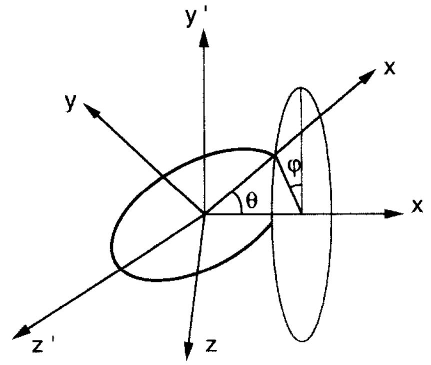

FIG. 8. Coordinate system for a fiber modelled as a prolate ellipsoid.

respectively. Then, averaging the conductivity tensor over angles θ through the distribution function $g(\theta)$ gives

$\overline{S_{ix}S_{jx}} = 0$ if $i \neq j$ and

$$\overline{S_{ix}S_{jx}} = \begin{cases} M = 1/(2m+3) & \text{if } i=j=y', z' \\ N = (2m+1)/(2m+3) & \text{if } i=j=x' \end{cases}. \quad \text{(A2)}$$

The resulting conductivity tensor is diagonal, with the components

$$K_e = 1+f_1\beta'_x(\alpha-1), \quad K_{e\perp} = 1+f_1\beta'(\alpha-1) \quad \text{(A3)}$$

where

$$\beta'_x = N\beta_x+(1-N)\beta, \quad \beta' = M\beta_x+(1-M)\beta. \quad \text{(A4)}$$

When $m = 0$ (randomly oriented fibers with elliptic cross section), $M = N = 1/3$, $\beta'_x = \beta' = (\beta_x+2\beta)/3$. When $m = \infty$ (parallel alignment) we get $N = 1$, $M = 0$, $\beta'_x = \beta_x$, $\beta' = \beta$. Then one finds the electric fields E_a and the dipole moments P_a for the directionally averaged ellipsoidal inclusion in an external field E_o

$$E_{ai} = \beta'_i E_{oi}, \quad P_{ai} = \beta'_i(\alpha-1)E_{oi}, \quad i = x, y, z. \quad \text{(A5)}$$

By ordinary effective medium theory [22] applied to the orthogonal directions x, y and z, the self-consistent conditions to be satisfied are

$$\langle P_{ai}\rangle = 0, \quad i = x, y, z. \quad \text{(A6)}$$

That yields the following equations for the effective conductivities:

$$f_1\left[\frac{2M}{K_{e\parallel}+n(\alpha-K_{e\parallel})}+\frac{N}{K_{e\parallel}+n_x(\alpha-K_{e\parallel})}\right](\alpha-K_e) + f_2\frac{3(1-K_{e\parallel})}{2K_{e\parallel}+1} = 0 \quad \text{(A7)}$$

$$f_1\left[\frac{M+N}{K_{e\perp}+n(\alpha-K_{e\perp})}+\frac{M}{K_{e\perp}+n_x(\alpha-K_{e\perp})}\right](\alpha-K_{e\perp}) + f_2\frac{3(1-K_{e\perp})}{2K_{e\perp}+1} = 0. \quad \text{(A8)}$$

In the case of a very prolate ellipsoid ('fiber'), $n = 1/2$, $n_x = 0$, and we get

$$f_1(\alpha-K_{e\parallel})\left[\frac{4M}{\alpha+K_{e\parallel}}+\frac{N}{K_{e\parallel}}\right]+f_2\frac{3(1-K_e)}{2K_{e\parallel}+1} = 0 \quad \text{(A9)}$$

$$f_1(\alpha-K_{e\perp})\left[\frac{2M+2N}{\alpha+K_{e\perp}}+\frac{M}{K_{e\perp}}\right]+f_2\frac{3(1-K_{e\perp})}{2K_{e\perp}+1} = 0. \quad \text{(A10)}$$